Quiz #9; Tuesday, date: 03/20/2018
MATH 53 Multivariable Calculus with Stankova
Section #117; time: 5 – 6:30 pm
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1. The following extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

$$f(x,y,z) = x^3 + y^3 + z^3; \qquad x^2 + y^2 + z^2 = 1.$$

Solution. We will rewrite the constraint as

$$g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0.$$

So the gradients of f and g are

$$\nabla f = \langle 3x^2, 3y^2, 3z^2 \rangle,$$
$$\nabla g = \langle 2x, 2y, 2z \rangle.$$

Using λ as the Lagrange multiplier, we want to solve for points such that

$$\nabla f = \lambda \nabla g$$
$$\langle 3x^2, 3y^2, 3z^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle,$$

so we have

$$x = 0 \text{ or } \frac{2\lambda}{3}, \qquad y = 0 \text{ or } \frac{2\lambda}{3}, \qquad z = 0 \text{ or } \frac{2\lambda}{3}.$$

Clearly not all of x, y, z cannot all be 0 because of the constraint that $x^2 + y^2 + z^2 = 1$. For the nonzero variables, they must all be the same. So the critical points are

$$(x,y,z) = \begin{cases} \pm(1,0,0) \text{ and their permutations} & \text{if one variable is nonzero} \\ \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \text{ and their permutations} & \text{if two variables are nonzero} \\ \pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) & \text{if all variables are nonzero} \end{cases}$$

Now we can plug all of these into the function, which gives us the values of ± 1, $\pm\frac{1}{\sqrt{2}}$, $\pm\frac{1}{\sqrt{3}}$. Hence the maximum value is 1 and the minimum value is -1.

2. *True / False?* If f is a continuous function such that $f(x,y) = -f(y,x)$, then

$$\int_a^b \int_a^b f(x,y)\,dx\,dy = 0.$$

Solution. **True.** We have

$$\int_a^b \int_a^b f(x,y)\,dx\,dy = -\int_a^b \int_a^b f(y,x)\,dx\,dy$$
$$= -\int_a^b \int_a^b f(y,x)\,dy\,dx$$
$$= -\int_a^b \int_a^b f(x,y)\,dx\,dy,$$

where the second equal sign follows from Fubini's theorem. Furthermore, intuitively the function is antisymmetric about the line $y = x$. Since the region on which we are integrating is symmetric about the line $y = x$, this should follow from symmetry.

3. *True / False?* For a continuous function f, suppose $f_{\max}, f_{\min}, f_{\text{avg}}$ are its absolute maximum, absolute minimum and average value on a rectangle. Then we must have

$$f_{\max} \geq f_{\text{avg}} \geq f_{\min}$$

Solution. **True.** We have $f_{\max} \geq f(x) \geq f_{\min}$. Now $f_{\max} - f(x)$ and $f(x) - f_{\min}$ are both nonnegative functions, so their average values must be nonnegative as well, which are $f_{\max} - f_{\text{avg}}$ and $f_{\text{avg}} - f_{\min}$. So $f_{\max} \geq f_{\text{avg}} \geq f_{\min}$.